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COLLAPSE OF A SPHERICAL CAVITY IN A MEDIUM COMPLETELY TRANSPARENT TO VOLUME RADIATION

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When a spherical cavity collapses there develop in the vicinity of the center a number of unique features in flow gas dynamic characteristics, which can essentially be described by a self-similar solution corresponding to the process under consideration.

Self-similar solutions for gas dynamic flows upon collapse of a spherical cavity have been found using the assumption of flow isoentropicity [1]. In the present study the flow will be considered with presence of radiation losses in the medium, which is completely transparent to volume radiation, these losses developing when the gas temperature outside the cavity is sufficiently high. It will be assumed that the character of the radiation corresponds to a braking mechanism of free—free electron transitions, since at sufficiently high temperature all atoms of the material are completely ionized. The gas dynamic equations, the self-similar solution of which will be obtained below, differ from the classical system only in the presence of a term corresponding to radiant losses in the energy equation [2]:

$$\frac{\partial \rho\left(e+\frac{u^2}{2}\right)r^2}{\partial t}+\frac{\partial \rho u\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)r^2}{\partial r}=Q_0 r^2\rho^{\alpha}T^{\beta}$$

(where the constant $Q_0 < 0$, $\alpha = 2$, $\beta = 1/2$). Nevertheless this addition significantly changes the character of the flow: it becomes nonisoentropic; the self-similarity index increases as compared to the index obtained without consideration of radiant losses, which intensifies cumulation. The self-similar solution will be determined using the principles of [1], but will be significantly more complicated in view of the absence of an adiabatic interval in the case under consideration.

1. Mathematical Formulation of the Problem. In the self-similar solution the equation of state for the gas is assumed polytropic:

$$p=\rho c^2/\varkappa,\ e=p/[(\varkappa-1)\rho],\ T=p/(R\rho)=c^2/(\varkappa R).$$

Here p is pressure; e is specific internal energy; ρ is density; T, temperature; c, speed of sound; R, universal gas constant; $\varkappa$, polytropy index.

For collapse of a spherical cavity r and t are the distance from the center of the cavity and the time measured from the moment of collapse ($t < 0$). After transition to dimensionless variables

$$t=t_0 t,\quad r=r_0 r,\quad c^2=r_0^2c^2/t_0^2,\quad u=r_0u/t_0,\quad \rho=\rho_0\rho,\quad p=\rho_0 r_0^2 p/t_0^2,$$

where any two quantities (for example, ρ_0 and t_0) are arbitrary positive constants with appropriate dimensions, and

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$$r_0 = \left[\frac{\varkappa(\varkappa-1)Q_0}{(\varkappa R)^\beta}\right]^{1/2(1-\beta)} t_0^{(3-2\beta)/2(1-\beta)} \rho_0^{(\alpha-1)/2(1-\beta)},$$

the equations describing gas dynamic processes in a medium completely transparent to volume radiation for the case of spherical symmetry have the form

$$\frac{1}{p}\frac{\partial p}{\partial t} - \frac{1}{c^2}\frac{\partial c^2}{\partial t} + u\left(\frac{1}{p}\frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{\partial c^2}{\partial r}\right) + \frac{\partial u}{\partial r} + \frac{2u}{r} = 0,$$

$$\frac{\varkappa}{c^2}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) + \frac{1}{p}\frac{\partial p}{\partial r} = 0, \tag{1.1}$$

$$\frac{1}{c^2}\left(\frac{\partial c^2}{\partial t} + u\frac{\partial c^2}{\partial r}\right) + (\varkappa-1)\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \varkappa^{\alpha-1}p^{\alpha-1}c^{2(\beta-\alpha)} = 0.$$

The solution to be found must satisfy the following conditions: on the cavity boundary, which is a free surface,

$$dr/dt = u,\ p = 0,\ c^2 = 0; \tag{1.2}$$

at the center ($r = 0$, $t > 0$)

$$u(0,\ t) = 0. \tag{1.3}$$

As $r \to \infty$ the pressure and speed of sound must be finite.

The problem posed admits a self-similar solution. System (1.1) with boundary conditions (1.2) and (1.3) permit a group of transforms

$$r \to ar,\ t \to a^k t,\ u \to a^{1-k}u,\ c^2 \to a^{2(1-k)}c^2,\ p \to a^{\frac{2(1-k)(\alpha-\beta)-k}{\alpha-1}} p$$

for any exponent k. This allows us to seek the self-similar solution in the form

$$u = \frac{r}{t}U(\xi),\quad c^2 = \frac{r^2}{t^2}F(\xi),\quad p = r^{2(\alpha-\beta)/(\alpha-1)}|t|^{[2(\beta-\alpha)-1]/(\alpha-1)}P(\xi)\quad (\xi = \xi_0 r^{-k}t). \tag{1.4}$$

Substituting Eq. (1.4) in Eq. (1.1), we obtain a system of ordinary differential equations for the self-similar representations $F(\xi)$, $U(\xi)$, and $P(\xi)$:

$$(1-kU)\left(\frac{F'}{F} - \frac{P'}{P}\right)\xi - kU'\xi + \frac{3\alpha-2\beta-1}{\alpha-1}U - \frac{3-2\beta}{\alpha-1} = 0; \tag{1.5a}$$

$$\frac{\varkappa(1-kU)}{F}U'\xi - k\frac{P'}{P}\xi + \varkappa\frac{U^2-U}{F} + \frac{2(\alpha-\beta)}{\alpha-1} = 0; \tag{1.5b}$$

$$(1-kU)\frac{F'}{F}\xi - (\varkappa-1)kU'\xi + (3\varkappa-1)U + \varkappa^{\alpha-1}P^{\alpha-1}F^{\beta-\alpha} - 2 = 0. \tag{1.5c}$$

The boundary conditions for this system follow from Eqs. (1.2), (1.3). In view of the self-similarity, the cavity boundary corresponds to the line ξ = const. Without destroying generality, by an appropriate choice of ξ_0 we may make $\xi = 1$. Along the line ξ = const $dr/dt = r/kt$, so it follows from Eq. (1.2) that

$$\xi = 1\colon\ U(1) = 1/k,\ P = 0,\ F = 0. \tag{1.6}$$

The center ($r = 0$, $t > 0$) corresponds to the line $\xi = -\infty$, and in view of Eqs. (1.3), (1.4)

$$|U(\xi)\xi^{-1/k}| \to 0 \text{ as } \xi \to -\infty. \tag{1.7}$$

The focus section ($t = 0$, $r > 0$) corresponds to the line $\xi = 0$, along which the pressure, velocity, and speed of sound are functions of the radius alone. Therefore as $\xi \to 0$ the solution has the asymptote

$$P(\xi) \sim \xi^{[1+2(\alpha-\beta)]/(\alpha-1)},\ U(\xi) \sim \xi,\ F(\xi) \sim \xi^2, \tag{1.8}$$

whence as $r \to \infty$ in view of the requirement of finiteness of the functions $p(r,\ t)$, $u(r,\ t)$, $c^2(r,\ t)$ there follows the inequality $k \geq 1$.

Thus the problem under consideration reduces to finding the self-similarity index k ($k \geq 1$) at which a solution of system (1.5) satisfying conditions (1.6)-(1.9) exists, and obtaining that solution.

2. Determination of the Self-Similarity Index k. We write system (1.5) as two equations in the phase space (U, F, P)

$$\frac{dF}{dU} = F\frac{[\varkappa(1-kU)^2 - k^2F]\,b + (\varkappa-1)\,k\,[(1-kU)\,a + kdF]}{(1-kU)\,[(1-kU)\,a + k\,(b+d)\,F]},$$
$$\frac{dP}{dU} = P\frac{\varkappa\,[ka + (1-kU)\,(b+d)]}{(1-kU)\,a + k\,(b+d)\,F} \tag{2.1}$$

and the quadratures

$$\xi\frac{dU}{d\xi} = \frac{a\,(1-kU) + k\,(b+d)\,F}{\varkappa\,[k^2F - (1-kU)^2]}. \tag{2.2}$$

Here

$$a = \varkappa(U^2 - U),\ b = (3\varkappa - 1)U \pm \varkappa^{\alpha-1}P^{\alpha-1}F^{\beta-\alpha} - 2$$
$$(+ \text{ for } t < 0,\ - \text{ for } t > 0), \tag{2.3}$$
$$d = \frac{3\alpha - 2\beta - 1}{\alpha - 1}U - \frac{3 - 2\beta}{\alpha - 1}.$$

The initial data of Eq. (1.6) ($\xi = 1$) define the singular point A of system (2.1). The nontrivial solution departing from point A corresponds to the asymptote

$$F = F_0\,(1 - kU),\ P = P_0(1 - kU)^{(\alpha-\beta)/(\alpha-1)}$$
$$\left(F_0 = \frac{\varkappa\,(k-1)\,(\alpha-1)}{k^2\,[3\alpha - 2\beta - 1 - (3 - 2\beta)\,k]},\right. \tag{2.4}$$
$$\left.P_0 = \frac{1}{\varkappa}\left[\frac{\varkappa\,[\beta - 1 - k\,(3 - 2\beta)]}{(\alpha-\beta)\,k} + \frac{1}{k} + 2\right]^{1/(\alpha-1)} F_0^{(\alpha-\beta)/(\alpha-1)}\right).$$

According to Eqs. (2.2) and (2.4) as $\xi \to 1$

$$U = \frac{1}{k} + U_0\,(1-\xi)\quad \left(U_0 = \frac{1 - \beta + k\,(3 - 2\beta)}{(\alpha-\beta)\,k} - \frac{3\varkappa}{k} < 0\right). \tag{2.5}$$

Thus on the free surface p, ρ, T and the entropy function $S = p\rho^{-\varkappa}$ vanish. As ξ changes from 1 to 0 there must exist a value $\xi_1 (0 < \xi_1 < 1)$, such that

$$R = k^2F - (1 - kU)^2 = 0. \tag{2.6}$$

For $0 < 1 - \xi < \varepsilon$ (ε being sufficiently small) according to Eqs. (2.4) and (2.5), $R > 0$, and at $\xi = 0$, according to Eq. (1.8), $R < 0$. For the dependence of the gas dynamic quantities upon ξ to be unique it is necessary that at $\xi = \xi_1$

$$a(1 - kU) + k(b + d)F = 0. \tag{2.7}$$

It then follows from Eqs. (2.6) and (2.7) that the numerators of the right sides of system (2.1) vanish. The line $\xi = \xi_1$ in the plane (r, t) corresponds to the characteristic arriving at the center at the moment of focusing.

The equation of this characteristic and the relationships along it have the form

$$\frac{dr}{dt} = u - c,\quad \frac{1}{p}\frac{dp}{dt} - \frac{\varkappa}{c}\frac{du}{dt} = -\frac{2\varkappa u}{r} + \varkappa^{\alpha-1}p^{\alpha-1}c^{2(\beta-\alpha)}.$$

Hence, along the line ξ = const, which is a characteristic, the relationship $r/kt = u - c$ must be satisfied, or in self-similar variables $1 - k(U - \sqrt{F}) = 0$, i.e., the equality $R = 0$. After transformation to self-similar variables with consideration of $R = 0$ the relationship along the characteristic transforms to Eq. (2.7), and Eqs. (2.6), (2.7) define the singular line of Eq. (2.1), which must intersect the unknown integral curve in phase space at the singular point.

3. Integration in the Vicinity of the Singular Point. To eliminate fractional powers we introduce the variable

$$T = P^{\alpha-1}F^{\beta-\alpha}. \tag{3.1}$$

We then rewrite system (1.8) as

$$\frac{dF}{dU} = \frac{F}{1-kU}[(\varkappa-1)k + \varkappa b\Phi],$$

$$\frac{dT}{dU} = \frac{T}{1-kU}\{k[(\alpha-\beta)+\varkappa(b-1)] + \varkappa[(\beta-1)b + (\alpha-1)d]\Phi\}. \tag{3.2}$$

Here $\Phi = \frac{(1-kU)^2 - k^2F}{(1-kU)a + k(b+d)F}$; we obtain a, b, d from Eq. (2.3) with consideration of Eq. (3.1). The self-similarity index is found from the condition of existence of an integral curve departing from point A and intersecting the singular line defined by Eqs. (2.6), (2.7).

We write the equation of this line in explicit form

$$F = \frac{1}{k}(1-kU)^2, \quad T = \varkappa^{1-\alpha}\left\{-\frac{U}{1-kU}[2\varkappa(1-kU) - \varkappa(k-1)] + \frac{2(\alpha-\beta)(k-1)+k}{k(\alpha-1)}\right\}. \tag{3.3}$$

It follows from numerical calculations that the desired intersection is achieved for a whole range of k values at the singular point B, which is a function of k:

$$1.06 < k < 1.18. \tag{3.4}$$

The character of the singular point is determined by the roots of the characteristic polynomial, a general representation of which can be obtained in the following manner.

Let U_0, F_0, T_0 be the coordinates of the singular point. With the replacement of variables $U = U_0 + x$, $F = F_0 + f$, $T = T_0 + t$ in the vicinity of the point system (3.2) takes on the form

$$\frac{df}{dx} = f_0 + f_1\frac{a_1x + b_1f}{c_1x + d_1f + e_1t}, \quad \frac{dt}{dx} = t_0 + t_1\frac{a_1x + b_1t}{c_1x + d_1t + e_1t}.$$

The coefficients appearing therein are functions of the singular point coordinates. The system characteristic polynomial

$$\begin{vmatrix} f_0d_1 + b_1f_1 - \lambda & e_1f_0 & c_1f_0 + a_1f_1 \\ t_0d_1 + b_1t_1 & e_1t_0 - \lambda & c_1t_0 + a_1t_1 \\ d_1 & e_1 & c_1 - \lambda \end{vmatrix} = 0. \tag{3.5}$$

It can easily be seen that one of the roots of polynomial (3.5) is equal to zero. Thus, after a corresponding linear replacement of variables the phase space layers into two-dimensional planes in each of which the pattern of integral curves in the vicinity of the singular point is one and the same. The two other roots obtained for a series of k values from the range of Eq. (3.4) prove to be of the same sign. Thus, the character of the singular point B in the cases considered is a generalized node, and consequently the integral curves reach this point in a nonanalytic manner, i.e., a weak discontinuity develops. Since the line $\xi = \xi_1$, corresponding to point B is a characteristic of the system, a weak discontinuity is admissible thereon. However in the problem under consideration before the moment of collapse this characteristic is in no way physically distinguishable. Therefore in the given case the weak discontinuity is not justified, and our goal is to determine the value of k at which the passage of the integral curve through B is analytic.

To distinguish this analytical curve from other ones we will use a procedure from [1]. Let λ_1 and λ_2 be roots of the characteristic polynomial, $|\lambda_2| > |\lambda_1|$, by n in the vicinity of the point B we have the expansion

$$\begin{aligned} F &= F_0 + F_1(U-U_0) + \ldots + F_n(U-U_0)^n + C_F(U-U_0)^\delta + F_{n+1}(U-U_0)^{n+1} + \ldots, \\ T &= T_0 + T_1(U-U_0) + \ldots + T_n(U-U_0)^n + C_T(U-U_0)^\delta + T_{n+1}(U-U_0)^{n+1} + \ldots, \end{aligned} \tag{3.6}$$

where $n < \delta = \lambda_2/\lambda_1 < n + 1$; C_F is an arbitrary number; C_T is a function of C_F. The angular coefficients of the tangents to the integral curves at point B can have two values:

$$\begin{aligned} F_1 &= \frac{k^2D + CM - L}{2M} \pm \sqrt{\left(\frac{k^2D + CM - L}{2M}\right)^2 + \frac{LC - 2DV_0}{M}}, \\ T_1 &= \frac{AD - BC + BF_1}{D}. \end{aligned} \tag{3.7}$$

Here

$$L = a_0 + \frac{\varkappa V_0(2V_0 - 2 + k) - V_0^2\left(3\varkappa + 2\frac{\alpha-\beta}{\alpha-1} - k\varkappa^{\alpha-1}\left(A - \frac{B}{b}\right)\right)}{k^2},$$

$$M = \frac{2(\alpha-\beta)}{\alpha-1}V_0 - k(b_0 + d_0) + \frac{\varkappa^{\alpha-1}V_0^2 B}{k^2 D}, \quad A = \frac{T_0}{V_0}[\varkappa(1-\beta) + \beta - \alpha],$$

$$B = \frac{T_0}{V_0}\varkappa[(1-\beta)b_0 - (\alpha-1)d_0], \quad D = -\frac{(\varkappa-1)F_0 b_0}{kV_0}, \tag{3.8}$$

$$a_0 = \varkappa(U_0^2 - U_0) + 2\frac{\alpha-\beta}{\alpha-1}F_0,$$

$$b_0 = \frac{3\varkappa-1}{k}(V_0 - 1) + \varkappa^{\alpha-1}T_0 + 2, \quad d_0 = \frac{3\alpha - 2\beta - 1}{(\alpha-1)k}(V_0 - 1) + \frac{3-2\beta}{\alpha-1}, \quad V_0 = 1 - kU_0.$$

The desired integral curve belongs to the sheaf of curves which enter the node with a common tangent, the angular coefficient of which can be found from Eqs. (3.7), (3.8) with a plus sign before the radical. The coefficients F_k and T_k for $k \geq 2$ are determined uniquely as the solution of a system of linear equations obtained after substituting expansion (3.6) in system (3.2). We perform the replacement

$$y_1 = \frac{F - F_0 - F_1 x - \ldots - F_n x^n}{x^{n+1}} - F_{n+1},$$
$$y_2 = \frac{T - T_0 - T_1 x - \ldots - T_n x^n}{x^{n+1}} - T_{n+1}. \tag{3.9}$$

It is obvious that for all curves of the sheaf which have nonzero C_F and C_T at $x = 0$ the functions y_1 and y_2 increase without limit. According to the Briot—Bouquet theorem, within the sheaf of integral curves entering the node and having a common tangent at that point there must be one analytical curve (or an infinite number for a dicritical node) to which there correspond C_F and C_T values of zero. Thus for the desired analytical curve at $x = 0$ the functions y_1 and y_2 vanish, while

$$\left.\frac{dy_1}{dx}\right|_{x=0} = F_{n+2}, \quad \left.\frac{dy_2}{dx}\right|_{x=0} = T_{n+2}. \tag{3.10}$$

Thus, the procedure for calculating k reduces to the following operations. For a given k from the interval of Eq. (3.4) we find the point of intersection B of the integral curve exiting from point A in the direction of Eq. (2.4) with singular line Eq. (3.3). In the vicinity of B the coefficients of the expansions T_i and F_i are determined, in particular, that value of n for which $n < \delta < n + 1$. Performing the replacement of Eq. (3.9) in Eq. (3.2), we find the integral curve **exiting from the point** $x = 0$, $y_1 = 0$, $y_2 = 0$ in the direction of Eq. (3.10). The value of k will be the desired one if at $x = 1/k - U_0$ (the x-value corresponding to point A), according to Eq. (3.9), $F = 0$, $T = P_0^{\alpha-1}F_0^{\beta-\alpha}$, where P_0 and F_0 are defined by Eq. (2.4), i.e., by the specified initial data at point A. Calculation of the coefficients of the expansions T_i and F_i and transition to the functions y_1 and y_2 for $n > 1$ requires cumbersome computations, which were performed on a computer using the SANTRA symbol-analytic transform system [3].

After the replacement of Eq. (3.9) system (3.2) takes on the form

$$\frac{dy_i}{dx} = \frac{P_i}{x^3 R}, \tag{3.11}$$

where

$$P_i = \sum_{k=3}^{9}(A_k^i + B_k^i y_1 + C_k^i y_2 + D_k^i y_i^2 + G_k^i y_1 y_2)x^k + E_k^i y_i^2 y_{3-i} x^k;$$

$$R = \sum_{k=1}^{7}(L_k + M_k y_1 + N_k y_2 + Q_k y_1 y_2)x^k; \quad A_3^i = 0, \quad D_k^i = G_k^i = 0 \quad (k \leqslant 4);$$

$$E_k^i = 0 \quad (k \leqslant 7); \quad M_k = N_k = 0 \quad (k \leqslant 2); \quad Q_k = 0 \quad (k \leqslant 5).$$

The desired k value, corresponding to analytical passage of the integral curve through the point B at $\alpha = 2$, $\beta = 1/2$, $\varkappa = 5/3$ is equal to 1.090853, and at this point $U_B = 0.849530$, $F_B = 0.00451371$, $T_B = 0.103991$. We note that in the region $k = 1.090853$ $\delta \approx 2$, therefore in Eq. (3.9) $n = 2$.

4. Passage of the Integral Curve through the Point Corresponding to the Focus Section. Having defined k, we continue integration of system (3.11), insuring analytical departure from the node B, from the point B to the point 0, at which U = 0. Upon approach to the point 0 we have the asymptote

$$F \approx F_0 U^2,\ T \approx T_0 U. \tag{4.1}$$

It follows from an equation corresponding to quadrature (2.2) that the point of the phase space (U = 0, F = 0, T = 0) corresponds to the value $\xi = 0$, i.e., the focusing section t = 0. According to asymptote (4.1) and the definition of T we find that $P \sim U^{[1+2(\alpha-\beta)]/(\alpha-1)}$, and from quadrature (2.2), $U \sim \xi$. Thus, with consideration of Eq. (1.4) the distribution of the gas dynamic functions in the focus section t = 0 appears as: $u \sim r^{1-k}$, $c^2 \sim r^{2(1-k)}$, $p \sim r^{[2(\alpha-\beta)(1-k)-1]/(\alpha-1)}$. Upon continuing the integration of system (3.2) ($t > 0$, $\xi < 0$) the derivatives $U'(\xi)$, $F'(\xi)$, $T'(\xi)$ become infinite at some value of ξ^*, i.e., $U(\xi)$, $F(\xi)$, $T(\xi)$ cease being unambiguous functions of ξ. This indicates that no continuous self-similar solution exists joining the point 0 corresponding to the focus section to the point C (r = 0, t > 0) corresponding to the center, i.e., a shock wave reflected from the center develops, which one would expect from physical considerations.

5. Study of the Solution in the Vicinity of the Center (r = 0, t > 0). At the center (r = 0, t > 0) the velocity u(0, t) = 0, and the pressure p(0, t) and speed of sound c(0, t) must be finite functions of the time t. For self-similar representations the boundary conditions are:

$$\xi = -\infty,\ |U(\xi)\xi^{-1/k}| < \infty,\ P(\xi) \sim |\xi|^{2(\alpha-\beta)/(\alpha-1)k},$$
$$F(\xi) \sim |\xi|^{2/k}, \tag{5.1}$$

and by definition

$$T(\xi) \to T_0 = \mathrm{const},\ \xi \to -\infty. \tag{5.2}$$

These conditions are satisfied only by the final value $U(\xi) \to U_0$ $(0 < U_0 < \infty)$.

We assume that $U(\xi) \sim |\xi|^{\gamma}$. Then in light of Eq. (4.1) from Eq. (1.5c) we have for $\gamma > 0$ $(|U(\xi)| \to \infty)$ $-k\left(\frac{2(\alpha-\beta)}{(\alpha-1)k} - \frac{2}{k}\right) - k\gamma + \frac{3\alpha-2\beta-1}{\alpha-1} = 0$, hence, $\gamma = 3/k$, which contradicts the condition $|U(\xi)\xi^{-1/k}| < \infty$. We then assume that $\gamma < 0$ $(U(\xi) \to 0)$, hence $\left(\frac{2(\alpha-\beta)}{\alpha-1} - 2\right)\frac{1}{k} - \frac{3-2\beta}{\alpha-1} = 0$, and consequently, $k = 2(1-\beta)/(3-2\beta) < 1$, which contradicts $k > 1$.

Thus the value of U_0 must be finite and is found by substitution of Eq. (5.1) and $U = U_0$ in Eq. (1.5a): $U_0 = [(3-2\beta)k + 2(\beta-1)]/(\alpha-1) > 0$. We define the value of T_0 by substituting U_0 and Eqs. (5.1), (5.2) in Eq. (1.5c): $T_0 = \varkappa^{1-\alpha}[2/k + (3\varkappa-1)U_0 - 2] > 0$. Consequentquently, as $\xi \to -\infty$ $U \to U_0$, $T \to T_0$, $F \to F_0\xi^{2/k}$.

It is obvious that the behavior of the integral curves in the vicinity of this point can be studied conveniently by making the substitution

$$f = 1/F,\ U = U_0 + x,\ T = T_0 + y. \tag{5.3}$$

As a result we have

$$\frac{dy}{dx} = \frac{T}{1-kU}\{k[\alpha-\beta+\varkappa(\beta-1)] + \varkappa[(\beta-1)c_1 + (\alpha-1)d_1]R\},$$
$$\frac{df}{dx} = -\frac{f}{1-kU}\{(\varkappa-1)k + \varkappa cR\}. \tag{5.4}$$

Here $R = \dfrac{(1-kU)^2 f - k^2}{(1-kU)A_1 - kA_0 x + k(c_1 + d_1)}$;

$$A_0 = \frac{2(\alpha-\beta)}{\alpha-1},\quad A_1 = \varkappa f\left[U_0^2 - U_0 + (2U_0-1)x + x^2\right];$$
$$c = c_0 + c_1,\ c_0 = 2(U_0 - 1/k),\ c_1 = (3\varkappa-1)x - x^{\alpha-1}y;$$
$$d_0 = \frac{(3\alpha-2\beta-1)U_0 - (3-2\beta)}{\alpha-1},\quad d_1 = \frac{3\alpha-2\beta-1}{\alpha-1}x.$$

The point ($x = 0$, $y = 0$, $f = 0$) is a singular point of system (5.4). The roots of the characteristic polynomial in the vicinity of the point are determined by the expressions $\lambda_1 = -k(\alpha - \beta)\varkappa^{\alpha-1}z_0$, $\lambda_2 = -2k\varkappa$, $\lambda_3 = 3k\varkappa$, $z_0 = kT_0/(1 - kU_0)$ (for the parameter values used two roots are negative and one positive). Thus this point is of the generalized saddle type. The integral curves departing from this point either coincide with the saddle separatrix corresponding to the direction

$$y \approx y_0 x,\ f \approx f_0 x, \tag{5.5}$$

where $y_0 = \dfrac{5\varkappa[\alpha - \beta + \varkappa(\beta - 1)]z_0}{\varkappa^{\alpha}(\beta - 1)z_0 + 2\varkappa}$, $f_0 = \dfrac{5z_0[2\varkappa - (\alpha - \beta)\varkappa^{\alpha-1}z_0]}{T_0(U_0^2 - U_0)[\varkappa^{\alpha}(\beta - 1)z_0 + 2\varkappa]}$, or belongs to a sheaf of integral curves which have an asymptote as $x \to 0$

$$\begin{gathered} y \approx y_1 x + \ldots + y_n x^n + C_1 x^{\gamma},\ f \approx Cx^{\gamma} \\ (\gamma = 2\varkappa^{2-\alpha}/[(\alpha - \beta)z_0],\ z_0 = kT_0/(1 - kU_0)). \end{gathered} \tag{5.6}$$

For $\alpha = 2$, $\beta = 1/2$, $\varkappa = 5/3$ $\gamma = 20/9$, consequently $n = 2$; C is arbitrary, and y_1, y_2, C_1 are found from

$$y_1 = \varkappa^{1-\alpha}[3\varkappa + (\alpha - \beta)\varkappa^{\alpha-1}z_0],\quad y_2 = \frac{(\alpha - \beta)\varkappa^{\alpha-1}ky_1(z_0 + y_1)}{(1 - kU_0)[2(\alpha - \beta)\varkappa^{\alpha-1}z_0 + 3\varkappa]},$$

$$C_1 = \frac{1}{5}C\frac{\varkappa}{k}(U_0^2 - U_0)[3\varkappa^{1-\alpha}(1 - kU_0) - k(\beta - 1)T_0].$$

We write the quadrature defining the dependence of U, F, T upon ξ in the form

$$\xi\frac{dx}{d\xi} = \frac{(1 - kU)A_1 - kA_0 x + k(c_1 + d_1)}{\varkappa[k^2 - f(1 - kU)^2]}. \tag{5.7}$$

By substituting Eqs. (5.5) and (5.6) in Eq. (5.7) for each of the integral curves we obtain the corresponding asymptote for the function $x(\xi)$ as $\xi \to -\infty$: for the saddle separatrix $x \sim |\xi|^{-2/k}$; for the integral curve belonging to the sheaf, $x \sim |\xi|^{-2/k\gamma}$. In either of these cases $F \sim |\xi|^{2/k}$, i.e., boundary conditions (5.1), (5.2) are satisfied.

Thus, the functions of time corresponding to the values of the gas dynamic quantities at the center can be expressed as

$$\begin{gathered} u(0, t) = 0,\ c^2(0, t) = F_0 t^{2(1-k)/k}, \\ p(0, t) = P_0 t^{[2(\alpha-\beta)(1-k)-k]/(\alpha-1)k},\ \rho(0, t) = R_0 t^{[2(1-\beta)(1-k)-k]/(\alpha-1)k}. \end{gathered}$$

6. Determination of the Reflected Shock Wave Front. It has been noted previously that a reflected shock wave develops behind the focus section. In view of self similarity the reflected shock wave front corresponds to a line $\xi = \xi_f = \text{const}$, so that the shock wave velocity $D = r/kt$. For self-similar representations of the gas dynamic functions the conditions on the shock wave front can be rewritten in the form

$$\begin{gathered} U_1 = \frac{1}{k} - \frac{(\varkappa - 1)L_0^2 + 2F_0}{(\varkappa + 1)L_0}, \\ F_1 = [\varkappa L_0(L_0 - L_1) + F_0]L_1/L_0, \\ T_1 = T_0(F/F_0)^{\beta-1}(L_0/L_1)^{\alpha-1}, \end{gathered} \tag{6.1}$$

where $L_0 = 1/k - U_0$; $L_1 = 1/k - U_1$; the subscript 0 refers to functions ahead of the front, and 1, to functions behind the front.

We will now define the shock wave front and the values of the gas dynamic functions thereon. For each integral curve departing from the point 0 in the direction $\xi < 0$, i.e., at $t > 0$, according to Eq. (6.1) the values of U_1, F_1, T_1 define the curve Φ_0 in the phase space. The shock wave front corresponds to the point of intersection of the curve Φ_0 with some integral curve departing from the center ($r = 0$, $t > 0$) following one of the asymptotes Eq. (5.5) or (5.6). One must obviously consider that the variables on this curve can be recalculated according to Eq. (5.3). With consideration of the three-dimensionality of the phase space it is improbable that the intersection point would lie on the saddle separatrix departing from the center (asymptote (5.5)), which was confirmed by calculations. The sheaf of integral curves departing from the center given by Eq. (5.6) forms a surface in the phase space as C changes ($0 < C < \infty$). The integral curve Φ_0 intersects this surface at a point lying on the curve of the sheaf corresponding to C = 1077. Thus a solution has been completely obtained in phase space. The dependence of the self-similar representations of the gas dynamic functions U, F, P on the self-similar variable ξ is defined ahead of the shock wave front by Eq. (2.2) and behind the front by Eq. (5.7).

Thus, the reflected shock wave front in the plane (r, t) corresponds to a line $\xi = \xi_f = -0.8809676$; ahead of the front $U_0 = -0.579504$, $F_0 = 0.181274$, $P_0 = 0.00869383$; behind the front $U_1 = 0.451876$, $F_1 = 0.304523$, $P_1 = 0.0524506$. The gas dynamic function distribution on the focus section is as follows: $u = 0.82318\ \xi_0 r^{1-k}$, $c^2 = 0.115196\ \xi_0^2 r^{2(1-k)}$, $p = 0.0463816\ \xi_0^4 r^{3-4k}$, $\rho = 0.671053\ \xi_0^2 r^{1-2k}$. The change with time in gas dynamic functions at the center is $u(0, t) = 0$, $p(0, t) = 1.524038 \cdot 10^{-6} |\xi_0|^{3/k} t^{3/k-4}$, $c^2(0, t) = 0.00028086\ |\xi_0|^{2/k} t^{2(1-k)/k}$, $\rho(0, t) = 0.009043\ |\xi_0|^{1/k}\ t^{1/k-2}$.

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BOUNDARY CONDITIONS ON A SHOCK WAVE IN A SUPERSONIC FLOW

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The theory in [1, 2] is used extensively in investigations of supersonic viscous gas flow around blunt bodies. A two-layer flow model consisting of a viscous shock layer and a domain of passage through the compression shock is proposed in these papers on the basis of an analysis of plane and axisymmetric supersonic flows around bodies.

The equations describing the domain of passage through the shock are integrated once and the relationships obtained (generalized Rankine—Hugoniot conditions) are used as boundary conditions on the outer boundary of the viscous shock layer. In contrast to the classical Rankine—Hugoniot conditions, the generalized conditions take account of molecular transport effects in the zone of the compression shock. The question of the influence of viscosity and heat conductivity on the flow of a homogeneous gas behind a strongly curved shockwave was first investigated in [3].

When chemical reactions are present in the flow, the problem of flow in a shock layer is already, in principle, not separated from the problem of the shockwave structure because of the presence of a source term in the mass conservation laws of the separate components. To compute it in the generalized Rankine—Hugoniot relationships the problem of the shockwave structure must be solved and joined with the solution within the shock layer. Avoiding this procedure to close the problem on the viscous shock layer, the chemical reactions within the shockwave are neglected by omitting the source term in the boundary conditions. As is shown in [4], let us note that the modified Rankine—Hugoniot relationships should be utilized for large Reynolds numbers; also sine application of the ordinary Rankine—Hugoniot relationships results in a finite error in the general case because of the origination of a source (sink) of the chemical component on the boundary.

The approximate analytic estimates executed in [5] showed that the two-layer model with the frozen wave front is justified for air for $V_\infty \lesssim 7$ km/sec. It is interesting to estimate numerically the influence of chemical reactions in the shock leading front on the flow characteristics.

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